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Stationary regimes for inventory processes

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Abstract

The inventory equation, $Z(t) = X(t) + L(t)$, where $X = \{X(t): t \geq 0\}$ is a given netput process and $\{L(t): t \geq 0\}$ is the corresponding lost potential process, is explored in the general case when X is a negative drift stochastic process that has asymptotically stationary increments. Our results show that if (as $s \to \infty$) $X_s \triangleq \{X(s+t) - X(s): t \geq 0\}$ converges in some sense to a process X^* with stationary increments and negative drift, then, regardless of initial conditions, $\theta_s Z \triangleq \{Z(s+t): t \geq 0\}$ converges in the same sense to a stationary version Z^*. We use coupling and shift-coupling methods and cover the cases of convergence in total variation and in total variation in mean, as well as strong convergence in mean. Our approach simplifies and extends the analysis of Borovkov (1976). We remark upon an application in regenerative process theory.

Keywords: Inventory equation; Coupling; Shift-coupling; Asymptotic stationarity; Convergence; Stationary distribution

1. Introduction

The classic inventory equation is given by

$$Z(t) = Z(0) + X(t) + L(t), \quad t \geq 0, \tag{1.1}$$

where $X(t)$ is a given function called the *netput process* with initial value $X(0) = 0$, $Z(0) \geq 0$ is the initial inventory and

$$L(t) \triangleq \left(\sup_{0 \leq s \leq t} - Z(0) - X(s) \right)^{+} \tag{1.2}$$

is called the *lost potential output process* (e.g., Harrison, 1985). $Z(t)$ represents the inventory level at time t. In many real applications X has the representation

$$X(t) = A(t) - B(t), \tag{1.3}$$

where $A(t)$ and $B(t)$ (both assumed to be nonnegative nondecreasing, with $A(0) = B(0) = 0$) are the cumulative *input* and *potential output* (respectively) during the time interval

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$[0,t]$. Besides the many inventory applications, (1.3) covers many queueing models, where $Z(t)$ could represent queue length or workload, for example. When Z is the queue length, then $A(t)$ denotes the number of arrivals by time t, while $B(t)$ denotes the number of potential services by time t. On the other hand, if Z is the workload, then $A(t)$ is the amount of work that has entered the system by t, while $B(t)$ is just t, the potential amount of time the server is busy. (1.3) also covers dam models where $Z(t)$ denotes water level. Of course, (1.3) does not allow for the possibility of X being a Brownian motion, but the more general set-up of (1.1) does. It is the BM case that is the main concern of Harrison (1985), in which case Z is called *reflected* (or *regulated*) Brownian motion and plays a major role in heavy-traffic limit theorems for queues (e.g., Kingman, 1961; Iglehart, 1965; Iglehart and Whitt, 1970). Further applications in the context of Levy processes can also be found in the literature (see, e.g., Kella and Whitt, 1991; Bardhan and Sigman, 1993). Sigman and Yao (1994) derive sufficient conditions to ensure finite moments for the steady-state distribution of (1.1).

It can be shown that Z in (1.1) satisfies the following recursive relationship:

$$Z(t) = Z(s) + X(t) - X(s) + \sup_{s \le u \le t} \big(- Z(s) - X(u) + X(s)\big)^+, \quad 0 \le s \le t < \infty. \tag{1.4}$$

This relationship will prove important in our analysis.

Borovkov (1976, Section 6) considered (1.1) in the context of workload for single server queues and X is any process with strictly stationary increments defined on all of the real line. He showed how to construct a stationary process Z^* that satisfies (1.1) and has a distribution to which, regardless of initial conditions, the shifted processes, $\theta_s Z = \{Z(s+t) : t \in \mathbb{R}\}$, converge. Borovkov thus did for Z what Loynes (1962) did in discrete time for the delay sequence $\{D_n\}$ of a single server FIFO queue, defined by the recurrence

$$D_{n+1} = (D_n + X_n)^+, \tag{1.5}$$

where X_n, assumed by Loynes to define a stationary sequence, is the nth customer's service time minus the next interarrival time.

Queuing models typically do not have stationary input, but have input that has a limiting (in some sense) stationary distribution (e.g., the input might be a marked point process that is periodic or, more generally, regenerative). Loynes' method can be applied to the stationary version of the input process to obtain a stationary distribution for the delay sequence, but this does not tell us whether the original queue, with the nonstationary input, has this limiting stationary distribution. Szczotka (1986) answered this question rigorously, by proving the following result: if the input to the queue is *asymptotically stationary* (in some sense) then so is the delay sequence (in the same sense), and, moreover, the limiting stationary distribution is the same as obtained by using Loynes' method on the limiting stationary version of input. Also see Rolski (1981, 1989).

The purpose of this paper is to prove similar convergences in continuous time for the inventory equation (1.1). We, however, employ coupling and shift-coupling methods (e.g., Aldous and Thorisson, 1993), which are more intuitive and greatly simplify the analysis. Sigman (1994), Ch. 6, has used this approach to provide direct proofs for convergence of the discrete-time delay sequence. We assume that the netput X is a stochastic process with paths in $D[0,\infty)$ (i.e., the set of functions $f : \mathbb{R}^+ \to \mathbb{R}$ that are right continuous with left hand limits, endowed with the Skorohod topology and the Borel field $\mathscr{B}(D[0,\infty))$, see Ethier and Kurtz (1986)). We show that if, as $s \to \infty$, the shifted processes $X_s \triangleq \{X(s+t) - X(s) : t \geq 0\}$ converge in some sense to a process X^* with stationary increments and negative drift, then regardless of initial conditions, the distribution of $\theta_s Z \triangleq \{Z(s+t) : t \geq 0\}$ converges, in the same sense, to the distribution of a stationary process Z^*. We consider the cases of convergence in total variation and total variation in mean, as well as strong convergence in mean. Here Z^* has the same distribution as is obtained by using Borovkov's construction on the stationary input process X^*. We do not require ergodicity of the netput process X.

In Section 2, we present some sample-path properties of the inventory process Z of (1.1). Section 3 introduces the techniques of coupling and shift-coupling and their relationship to the convergence of measures on D space. Section 4 presents our main result, namely, convergence of the shifted inventory process when the input process is asymptotically stationary. We also mention an application of our results to regenerative process theory.

2. Sample-path properties of the inventory process

The netput process X is assumed to be a real-valued stochastic process (on an underlying probability space $(\Omega, \mathscr{F}, P)$) with paths in $D[0,\infty)$. It is clear that the inventory process Z of (1.1) also has paths in $D[0,\infty)$. We will focus on some sample-path properties of the inventory process. These properties will form the basis of the convergence proofs to follow.

Lemma 2.1. (i) *Z is increasing in the initial inventory $Z(0)$, i.e., if we consider two versions of Z created from the same sample-path of netput X,*

$$\begin{aligned} Z'(t) &= Z'(0) + X(t) + \sup_{0\le u\le t} \left(- Z'(0) - X(u)\right)^+, \\ Z''(t) &= Z''(0) + X(t) + \sup_{0\le u\le t} \left(- Z''(0) - X(u)\right)^+, \end{aligned} \tag{2.1}$$

where $Z'(0) \geq Z''(0)$, then $Z'(t) \geq Z''(t)$, $\forall t \geq 0$.

(ii) *For every sample-path that satisfies $X(t)/t \to \lambda$, we have*

$$\lim_{t\to\infty} \frac{Z(t)}{t} = (\lambda)^+. \tag{2.2}$$

(iii) *For every sample path with $\lambda < 0$, there exists an increasing sequence of times $\{t_n\}_{n\geq 1}$ such that $Z(t_n) = 0$ and $t_n \to \infty$. For every sample path with $\lambda > 0$, $\lim_{t\to\infty} Z(t) = \infty$.*

(iv) *For any sample-path with $\lambda < 0$ and with initial inventory $Z(0) = z \geq 0$, let us create another version of the inventory process Z^0 with initial inventory $Z^0(0) = 0$. Then there exists a finite random time t_z such that $Z(t) = Z^0(t)$ for all $t \geq t_z$, i.e., the system is indistinguishable from having started initially empty. Indeed, $t_z = \inf\{t : Z(t) = 0\}$.*

Proof. (i) The result follows from the following inequality for any $t \geq 0$,

$$\begin{aligned} Z''(t) &= Z''(0) + X(t) + \sup_{0 \leq s \leq t} \left(-Z''(0) - X(s) \right)^+ \\ &\leq Z''(0) + X(t) + \sup_{0 \leq s \leq t} \Big(-Z''(0) - X(s) - \big(Z'(0) - Z''(0)\big) \Big)^+ \\ &\quad + \big(Z'(0) - Z''(0)\big) \\ &= Z'(0) + X(t) + \sup_{0 \leq s \leq t} \left(-Z'(0) - X(s) \right)^+ = Z'(t). \end{aligned} \tag{2.3}$$

(ii) For any sample path with $\lambda > 0$, we have $X(t) \to \infty$. Using the representation of (1.2), we know that $L(t)$ converges to something finite, which then gives us our result. Paths with $\lambda \leq 0$ require a little more work. Consider the perturbed netput process $X^{\varepsilon}(t) \triangleq X(t) + \varepsilon t$, where $\varepsilon > -\lambda$. Superscripting the perturbed system by ε, we have $\lambda^{\varepsilon} = \lambda + \varepsilon > 0$. Thus $Z^{\varepsilon}(t)/t \to \lambda + \varepsilon$. Now similar to (2.3), for any $t \geq 0$,

$$\begin{aligned} Z(t) &= Z(0) + X(t) + \sup_{0 \leq s \leq t} \left(-Z(0) - X(s) \right)^+ \\ &\leq Z(0) + X(t) + \sup_{0 \leq s \leq t} \left(-Z(0) - X(s) - \varepsilon s \right)^+ + \sup_{0 \leq s \leq t} \left(\varepsilon s \right)^+ \\ &= Z(0) + X^{\varepsilon}(t) + \sup_{0 \leq s \leq t} \left(-Z(0) - X^{\varepsilon}(s) \right)^+ = Z^{\varepsilon}(t). \end{aligned} \tag{2.4}$$

By taking limits as $t \to \infty$, we have

$$\limsup_{t \to \infty} \frac{Z(t)}{t} \leq \lambda + \varepsilon. \tag{2.5}$$

Since the right-hand side can be an arbitrarily small positive number, we have our result for the case of $\lambda \leq 0$.

(iii) Consider the lost potential output process L defined in (1.2). Clearly, when $\lambda < 0$,

$$L(t) = \Big(-\inf_{0 \leq s \leq t} Z(0) + X(s) \Big)^+ = t \Big(-\frac{Z(0)}{t} - \inf_{0 \leq s \leq t} \frac{X(s)}{t} \Big)^+ \underset{t \to \infty}{\longrightarrow} \infty. \tag{2.6}$$

Also, it is well-known and easy to show that L increases only when Z is at the origin (indeed, L is the *local time* of Z at 0, satisfying $\int_0^t 1_{\{Z(s) \geq 0\}} \mathrm{d}L(s) = 0$). Thus, Z must visit the origin infinitely often for a sequence of times increasing to ∞.

(iv) Using (i) and (iii), we know that at the time t_z, it must be true that $Z(t_z) = Z^0(t_z) = 0$. After this, the inputs to both systems are the same and (1.4) establishes the result. □

The parameter λ defined on each sample-path is essentially the average drift of the netput process X. Properties (ii) and (iii) in Lemma 2.1 state that if the netput has a positive long-run drift, then the inventory process will increase to infinity at the same rate, but if the drift is negative, then the system will empty infinitely often. Properties (i) and (iv) show that this is true for any level of initial inventory. In terms of the underlying probability space, Birkhoff's ergodic theorem (see, for example, Sigman, 1994, Ch. 2, Section 2.5) can be used to show $\lambda \equiv E[X(1)|\mathscr{I}]$, where $\mathscr{I}$ is the *invariant σ-field with respect to shifts.*

3. Convergence, coupling and the shifted processes

The previous section went over some sample-path properties. Our main interest, however, is to show convergence of measures defined on $\mathscr{B}(D[0,\infty))$, the Borel sets of $D[0,\infty)$. We consider three main modes of convergence. Consider a collection of probability measures $\{\mu_t : t \geq 0\}$ defined on $\mathscr{B}(D[0,\infty))$. The collection is said to converge to a probability measure μ in one of the following ways.

(1) *Strong convergence in mean*: If, for every Borel set $B \in \mathscr{B}(D[0,\infty))$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mu_s(B)\,\mathrm{d}s = \mu(B). \tag{3.1}$$

(2) *Convergence in variation*: If we have

$$\lim_{t\to\infty} \|\mu_t - \mu\| = 0. \tag{3.2}$$

(3) *Convergence in variation in mean*: If we have

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t \mu_s\,\mathrm{d}s - \mu \right\| = 0. \tag{3.3}$$

Here the $\|\cdot\|$ norm is defined as

$$\|\mu_1 - \mu_2\| = \sup_{B\in\mathscr{B}(D[0,\infty))} |\mu_1(B) - \mu_2(B)|. \tag{3.4}$$

We will relate these modes of convergence of measures to the the sample-path behaviour of processes that obey these laws.

For any real-valued process Y with paths in $D[0,\infty)$, define the shifted processes

$$\theta_s Y \triangleq \{Y(t+s);\ t \geq 0\}, \tag{3.5}$$

which are also processes in $D[0,\infty)$. With this, we now introduce the central tool in our analysis.

Definition. Two processes Y and Y^* are said to *admit coupling* at some random time T if there exist versions of Y and Y^* on a common probability space such that

$$Y(u+T) = Y^*(u+T), \quad u \geq 0; \qquad \text{i.e., } \theta_T Y \equiv \theta_T Y^*. \tag{3.6}$$

The processes are said to *admit shift-coupling* at random times T_1 and T_2 if there exist versions of Y and Y^* on a common probability space such that

$$Y(u+T_1) = Y^*(u+T_2), \quad u \geq 0; \qquad \text{i.e., } \theta_{T_1} Y \equiv \theta_{T_2} Y^*. \tag{3.7}$$

Though we refer to coupling of processes, it is really a property of the measures that control the processes. Indeed, as the next result shows, coupling of the processes is inherently linked to the convergence of the associated measures (see Thorisson (1994) for details, in particular the equivalence of (a′)–(c′)). We use $\mathscr{L}(Y)$ to denote the law of the process Y.

Lemma 3.1. *The following statements are equivalent:*

(1) *Two processes Y and Y^* admit coupling (shift-coupling) at some finite random time T (times T_1 and T_2).*

(2)

$$\lim_{t\to\infty} \|\mathscr{L}(Y_t) - \mathscr{L}(Y_t^*)\| = 0$$

$$\left(\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t \mathscr{L}(Y_s)\,\mathrm{d}s - \frac{1}{t}\int_0^t \mathscr{L}(Y_s^*)\,\mathrm{d}s \right\| = 0\right). \tag{3.8}$$

(3) *The laws of Y and Y^* agree when restricted to the tail σ-field (invariant σ-field).*

In particular, if Y^* is a stationary process, then the distributions of the shifted processes Y_s converge to the distribution of Y^* in total variation (total variation in mean).

This equivalence between coupling and convergence provides a powerful method for establishing convergence of measures. There is also a surprising equivalence between strong convergence in mean and total variation convergence in mean for the measures of shifted processes, as the following corollary shows.

Corollary 3.2. *For any two processes Y and Y^*,*

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t}\int_0^t P(Y_s \in B)\,\mathrm{d}s = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(Y_s^* \in B)\,\mathrm{d}s \quad \forall B \in \mathscr{B}(D[0,\infty)) \\ &\iff \lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t \mathscr{L}(Y_s)\,\mathrm{d}s - \frac{1}{t}\int_0^t \mathscr{L}(Y_s^*)\,\mathrm{d}s \right\| = 0. \end{aligned} \tag{3.9}$$

In other words, the shifted processes converge strongly in mean if and only if they converge in total variation in mean.

Proof. Clearly, total variation convergence in mean implies strong convergence in mean, so we need to only show the opposite direction. To this end, notice that strong convergence in mean implies convergence for every set $B \in \mathscr{B}(D[0,\infty))$, in particular, for every set in the invariant σ-field $\mathscr{I}$. For such sets, however, $P(Y_s \in B) = P(Y \in B)$, so that

$$P(Y \in B) = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(Y_s \in B)\,\mathrm{d}s = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(Y_s^* \in B)\,\mathrm{d}s = P(Y^* \in B). \tag{3.10}$$

Thus, the laws of Y and Y^* agree on the invariant σ-field, and Lemma 3.1(3) completes the argument. □

Note that this equivalence between the modes of convergence holds only when the sequence of measures are defined from shifting the processes, but not for any general sequence of measures.

We will also need the following result.

Lemma 3.3. *If a process Y admits coupling with a process Y' and Y' admits coupling with another process Y'', then Y admits coupling with Y''. Similarly, if Y admits shift-coupling with Y' and Y' admits shift-coupling with Y'', then Y admits shift-coupling with Y''.*

Lemma 3.3 follows immediately from Lemma 3.1(3) because the laws of all three processes agree when restricted to the tail σ-field (invariant σ-field).

The tools of coupling and shift-coupling will be our vehicles for proving convergence of the shifted inventory processes $\theta_s Z$. We however have to modify the notion of shifts for the netput process X. It is evident from (1.1) and others that the properties of the system depend more on the *increments* of the netput than on the total levels. Thus, stationarity and convergence of measures will be defined in terms of the increments. To be more precise, we define the shifted netput processes as

$$X_s \triangleq \{X(t+s) - X(s);\ t \geq 0\}, \tag{3.11}$$

which are also processes in $D[0,\infty)$. All the properties of coupling and shift-coupling presented so far hold for this new definition of shifts. For example, two netput processes X and X^* are said to admit shift-coupling at random times T_1 and T_2 if there exist versions of X and X^* on a common probability space such that

$$X(u+T_1) - X(T_1) = X^*(u+T_2) - X^*(T_2), \quad u \geq 0; \quad \text{i.e., } X_{T_1} \equiv X^*_{T_2}. \tag{3.12}$$

Lemmas 3.1 and 3.3 and Corollary 3.2 hold for these shifted processes as well. Moreover, the law of a netput process is invariant under shifts if and only if it has stationary increments.

4. Convergence by coupling

We begin with the definition of asymptotic stationarity in our set-up.

Definition. A netput process X is said to have asymptotically stationary increments if the probability measures $\mu_s \triangleq P(X_s \in \cdot)$ of the shifted processes, defined on the Borel sets of $D[0,\infty)$, converge in total variation, or in total variation in mean, to a probability measure μ on $D[0,\infty)$, as $s \to \infty$. Letting X^* denote a process with distribution μ, it is easily seen that X^* has stationary increments and satisfies $X^*(0) = 0$. We call X^* the *stationary increment version* of X.

The process X^* is a stationary increment process defined on $[0,\infty)$. In our analysis, we will require a stationary increment process $\tilde{X}^*$ defined on $(-\infty,\infty)$, whose distribution, when restricted to $[0,\infty)$, matches that of X^*. The existence of such a two-sided version is relatively straightforward to prove using Kolmogorov's criterion and some additional work (e.g., Borovkov, 1976), so we will assume its existence.

Let us review Borovkov's construction of a stationary version of $\{Z(t);\ t \geq 0\}$ for a stationary increment netput process (Borovkov, 1976, Section 6). Define the process

$$Z^*(t) = \sup_{u \leq t}(\tilde{X}^*(t) - \tilde{X}^*(u)). \tag{4.1}$$

It can be shown that Z^* is stationary and satisfies the evolution equation (1.4) with netput X^*.

Consider an inventory system with stationary increment netput X^*, and $X^*(t)/t \xrightarrow{\text{a.s.}} \alpha < 0$, and the initial inventory $Z(0) = z$, a constant. Borovkov has shown that the finite-dimensional distributions of the shifted inventory processes $\theta_s Z$ converge to the finite-dimensional distributions of Z^*. In fact, he shows that

$$P\{\sup_{u\geq 0}|\theta_t Z(u) - \theta_t Z^*(u)| \neq 0\} \to 0, \tag{4.2}$$

for $t \to \infty$. Borovkov also considers a case when the input process X is asymptotically stationary. He proves weak convergence of finite-dimensional distributions of $\theta_t Z$.

We now state our main result.

Theorem 4.1. *Let X converge asymptotically to a stationary increment process X^* in total variation (total variation in mean). Moreover, let X have negative drift, almost surely. Then, for any deterministic initial condition, $Z(0) = z \geq 0$, the distribution of $\theta_s Z$, converges in total variation (total variation in mean) to the distribution of the stationary process Z^*.*

The way we prove convergence is by showing that Z actually couples or shift-couples with Z^*.

Proof of Theorem 4.1. We show the proof for shift-coupling. Since $X \xrightarrow{\text{tvm}} X^*$, there is a common probability space which supports versions of X and X^* such that $X_{T_1} \equiv X^*_{T_2}$, a.s., for some finite random times T_1 and T_2. With these versions on the common probability space, we define the inventory process Z for netput X and the process inventory process $\hat{Z}$ for netput X^*, using (1.1). Using the representation of (1.4), we can rewrite the shifted inventory processes as

$$\begin{aligned} \theta_{T_1} Z(t) &= Z(T_1) + X_{T_1}(t) + \Big(\sup_{0\leq s\leq t}\big(-Z(T_1) - X_{T_1}(s)\big)\Big)^+; \\ \theta_{T_2} \hat{Z}(t) &= \hat{Z}(T_2) + X^*_{T_2}(t) + \Big(\sup_{0\leq s\leq t}\big(-\hat{Z}(T_2) - X^*_{T_2}(s)\big)\Big)^+. \end{aligned} \tag{4.3}$$

The inventory processes seen at these random times now differ only in the initial inventory, $Z(T_1)$ versus $\hat{Z}(T_2)$, since the incremental input processes are the same. By Lemma 2.1(iv), we know that there is a finite random time T' such that $\theta_{T_1} Z(T' + s) = \theta_{T_2} \hat{Z}(T' + s)$, $s \geq 0$. The time T' is given by $\inf\{t: \max\{\theta_{T_1} Z(t), \theta_{T_2} \hat{Z}(t)\} = 0\}$. This case is slightly different since we consider each system shifted to a different point in time, but (1.4) ensures that property (iv) of Lemma 2.1 holds here, too. Thus, Z and $\hat{Z}$ admit shift-coupling at the times $T_1 + T'$ and $T_2' + T'$, which is half the argument.

The other half is to show that $\hat{Z}$ and Z^* admit shift-coupling and then use Lemma 3.3. Consider a probability space which supports a version of the two-sided extension of the stationary netput, $\tilde{X}^*$, constructed from Lemma 3.2. On this space define the stationary inventory process Z^* using (4.1) and the process $\hat{Z}$ from the netput $\tilde{X}^*|_{[0,\infty)} \equiv X^*$. Now, comparing (1.1) and the definition of $\hat{Z}$, it is clear that the two systems differ only in their initial conditions, i.e., $Z^*(0)$ versus $\hat{Z}(0)$. Lemma 2.1(iv) then tells us that there is a time after which the processes are indistinguishable on every sample path. This implies that the original $\hat{Z}$ and Z^* admit shift-coupling. Invoking Lemma 3.3, it must be true that Z

shift-couples with Z^*. Then Lemma 3.1 implies that the shifted processes Z_s converge to Z^* in total variation in mean.

For the case of total variation convergence, we follow the same arguments with $T_1 = T_2 = T$. □

By Corollary 3.2, we get the following result for free.

Corollary 4.2. *Let X converge asymptotically to a stationary increment process X^* strongly in mean. Moreover, let X have negative drift, almost surely. Then, for any initial condition, $Z(0) \geq 0$, the distribution of $\theta_s Z$, converges strongly in mean to the distribution of the stationary process Z^*.*

Remark. An area of application of our results is in regenerative theory. Let $X(t)$ have regenerative increments in the sense that there exists a positive recurrent embedded renewal process with times $\tau_1 < \tau_2 < \tau_3 \cdots$, etc., such that $\{X(t+\tau_k)-X(\tau_k): t \geq 0;\ \{\tau_j\}_{j\geq k+1}\}$ has the same distribution for all k, and is independent of the past $\{X(t): t < \tau_k; \{\tau_j\}_{j\leq k}\}$. Then it can be shown that $X_s \xrightarrow{\text{tvm}} X^*$, for some stationary process X^*. Hence, if $EX^*(1) < 0$, or in other words, $E[X(\tau_2)-X(\tau_1)]/E[\tau_2-\tau_1] < 0$, then by the results of Section 4, we know that $\theta_s Z \xrightarrow{\text{tvm}} Z^*$. Furthermore, it now can be shown that Z is a one-dependent regenerative process, by using Harris recurrent Markov process (HRMP) techniques developed in Sigman (1990). In fact, X can be allowed more generally to be *governed by a HRMP*, i.e., $X_s = f(\Theta(s))$, where $\Theta(s)$ is a HRMP, and f is some mapping from the state-space of Θ to $D[0,\infty)$, analogous to section 6 in Sigman (1990).

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